CAPÍTULO 3

Exercícios 3.1

1. ***a***) $\iint_A f(x, y)\, dx\, dy = \int_0^1 \int_1^2 (x + 2y)\, dx\, dy = \int_0^1 \alpha(y)\, dy$ onde $\alpha(y) = \int_1^2 (x + 2y)\, dx =$

$$= \left[\frac{x^2}{2} + 2xy\right]_1^2 = 2y + \frac{3}{2}.$$

Então, $\iint_A f(x + 2y)\, dx\, dy = \int_0^1 \left[\int_1^2 (x + 2y)\, dx\right] dy =$

$$= \int_0^1 \left(2y + \frac{3}{2}\right) dy = \left[y^2 + \frac{3}{2}y\right]_0^1 = \frac{5}{2}.$$

Invertendo a ordem de integração,

$$\int_1^2 \left[\int_0^1 (x + 2y)\, dy\right] dx = \int_1^2 \left[(xy + y^2)\right]_0^1 dx =$$

$$= \int_1^2 (x + 1)\, dx = \left[\frac{x^2}{2} + x\right]_1^2 = \frac{5}{2}.$$

c) $\iint_A f(x, y)\, dx\, dy = \int_0^1 \int_1^2 \sqrt{x + y}\, dx\, dy =$

$$= \int_1^2 \left[\frac{2}{3}(x + y)^{3/2}\right]_0^1 dy = \frac{2}{3}\int_0^1 \left[(2 + y)^{3/2} - (1 + y)^{3/2}\right] dy =$$

$$= \frac{2}{3} \cdot \frac{2}{5}\left[(2 + y)^{5/2} - (1 + y)^{5/2}\right]_0^1 = \frac{4}{15}\left[3^{5/2} - 2^{5/2} - 2^{5/2} + 1\right] =$$

$$= \frac{4}{15}(9\sqrt{3} - 8\sqrt{2} + 1).$$

e) $\int_0^1 \int_1^2 dx\, dy = \int_0^1 [x]_1^2\, dy = \int_0^1 dy = [y]_0^1 = 1.$

g) $\int_0^1 \left[\int_1^2 y \cos xy \, dx \right] dy = \int_0^1 [\text{sen } xy]_1^2 \, dy =$

$= \int_0^1 (\text{sen } 2y - \text{sen } y) \, dy = \left[\frac{1}{2} (-\cos 2y) + \cos y \right]_0^1 =$

$= -\frac{1}{2} \cos 2 + \cos 1 + \frac{1}{2} - 1 = \cos 1 - \frac{1}{2} [1 + \cos 2].$

***i*)** $\int_0^1 \left[\int_1^2 y \, e^{xy} \, dx \right] dy = \int_0^1 \left[e^{xy} \right]_1^2 \, dy =$

$= \int_0^1 (e^{2y} - e^y) \, dy = \left[\frac{1}{2} e^{2y} - e^y \right]_0^1 =$

$= \frac{1}{2} e^2 - e - \frac{1}{2} + 1 = \frac{1}{2} (1 + e^2) - e.$

***l*)** $\int_0^1 \left[\int_1^2 x \text{ sen } \pi y \, dx \right] dy = \int_0^1 \left[(\text{sen } \pi y) \cdot \frac{x^2}{2} \right]_1^2 \, dy =$

$= \int_0^1 \left(2 \text{ sen } \pi y - \frac{1}{2} \text{ sen } \pi y \right) dy = \frac{3}{2} \int_0^1 \text{sen } \pi y \, dy =$

$= \frac{3}{2\pi} [\cos \pi y]_0^1 = \frac{3}{2\pi} [-\cos \pi - (-\cos 0)] = \frac{3}{2\pi} \cdot 2 = \frac{3}{\pi}.$

2. Temos

$$\iint_A f(x) \, g(y) \, dx \, dy = \int_c^d \int_a^b f(x) \, g(y) \, dx \, dy =$$

$$= \int_c^d \left[\int_a^b f(x) \, g(y) \, dx \right] dy = \int_c^d g(y) \left[\int_a^b f(x) \, dx \right] dy =$$

$$= \left[\int_a^b f(x) \, dx \right] \cdot \left[\int_c^d g(y) \, dy \right].$$

3. *a*) $\iint_A xy^2 \, dx \, dy = \int_2^3 \int_1^2 xy^2 \, dx \, dy =$

$= \left(\int_1^2 x \, dx \right) \cdot \left(\int_2^3 y^2 \, dy \right) = \left[\frac{x^2}{2} \right]_1^2 \cdot \left[\frac{y^3}{3} \right]_2^3 =$

$= \left(\frac{3}{2} \right) \cdot \left(\frac{19}{3} \right) = \frac{19}{2}.$

b) $\iint_A x \cos 2y \, dx \, dy = \left(\int_0^1 x \, dx\right) \cdot \left(\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos 2y \, dy\right) =$

$= \left[\frac{x^2}{2}\right]_0^1 \cdot \left[\frac{1}{2} \text{ sen } 2y\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}.$

c) $\iint_A x \ln y \, dx \, dy = \left(\int_0^2 x \, dx\right)\left(\int_1^2 \ln y \, dy\right) =$

$= \left[\frac{x^2}{2}\right]_0^2 \cdot [y \ln y - y]_1^2 = 2(2 \ln 2 - 1).$

e) $\iint_A \frac{\text{sen}^2 x}{1 + 4y^2} \, dx \, dy = \left(\int_0^{\frac{\pi}{2}} \text{sen}^2 x \, dx\right)\left(\int_0^{\frac{1}{2}} \frac{dy}{1 + 4y^2}\right) =$

$= \left[\frac{1}{2} \text{ sen } x \cos x + \frac{x}{2}\right]_0^{\frac{\pi}{2}} \left[\frac{1}{2} \text{ arctg } 2y\right]_0^{\frac{1}{2}} =$

$= \left(\frac{\pi}{4}\right) \cdot \left(\frac{\pi}{8}\right) = \frac{\pi^2}{32}.$

(Observe que para todo $n \geqslant 2$, $\int_0^{\frac{\pi}{2}} \text{sen}^n x \, dx = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \text{sen}^{n-2} \, dx$,

logo, $\int_0^{\frac{\pi}{2}} \text{sen}^2 x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{4}$.) (Veja Volume 1, Seção 12.3.)

4. a) $V = \iint_B (x + 2y) \, dx \, dy$, onde B é o retângulo $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Então,

$V = \int_0^1 \left[\int_0^1 (x + 2y) \, dx\right] dy = \int_0^1 \left[\frac{x^2}{2} + 2xy\right]_0^1 dy =$

$= \int_0^1 \left(\frac{1}{2} + 2y\right) dy = \left[\frac{y}{2} + y^2\right]_0^1 = \frac{3}{2}.$

c) $V = \iint_B xy \, e^{x^2 - y^2} \, dx \, dy$, onde B é o retângulo $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Então,

$V = \int_0^1 \int_0^1 xye^{x^2 - y^2} \, dx \, dy = \int_0^1 xe^{x^2} \, dx \int_0^1 ye^{-y^2} \, dy =$

$= \left[\frac{1}{2} e^{x^2}\right]_0^1 \left[-\frac{1}{2} e^{-y^2}\right]_0^1 = \frac{1}{4} (e - 1)(1 - e^{-1}).$

e) $V = \iint_B [(x + y + 2) - (x + y)]\, dx\, dy$ onde B é o retângulo $1 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 1$.
Então,

$$V = \iint_B 2\, dx\, dy = \int_0^1 \left[\int_1^2 2\, dx \right] dy =$$

$$= \int_0^1 [2x]_1^2\, dy = \int_0^1 2\, dy = [2y]_0^1 = 2.$$

f) $V = \iint_B (e^{x+y} - 1)\, dx\, dy$, onde B é o retângulo $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Então,

$V = \int_0^1 e^x\, dx \int_0^1 e^y\, dy - \int_0^1 dx \int_0^1 dy$, ou seja,

$V = \left[e^x\right]_0^1 \left[e^y\right]_0^1 - [x]_0^1\, [y]_0^1 = (e-1)^2 - 1$ e, portanto,

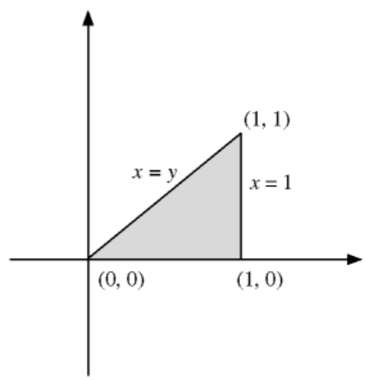
$V = e^2 - 2e.$

5. ***a***) $\iint_B y\, dx\, dy$, onde B é o triângulo de vértices (0, 0), (1, 0) e (1, 1).

$$\int_0^1 \int_{a(y)}^{b(y)} y\, dx\, dy = \int_0^1 \left[\int_y^1 y\, dx \right] dy =$$

$$= \int_0^1 [xy]_y^1\, dy = \int_0^1 (y - y^2)\, dy =$$

$$= \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \frac{1}{6}.$$



Invertendo a ordem de integração,

$$\iint_B y\, dy\, dx = \int_0^1 \left[\int_{c(x)}^{d(x)} y\, dy \right] dx =$$

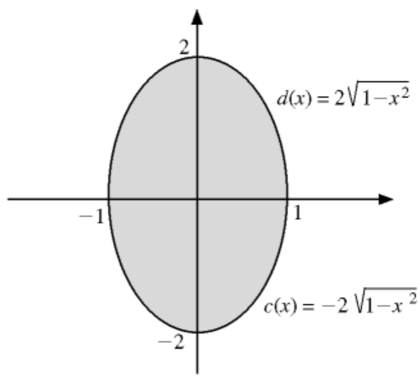
$$= \int_0^1 \left[\int_0^x y\, dy \right] dx = \int_0^1 \left[\frac{y^2}{2} \right]_0^x dx = \int_0^1 \frac{x}{2}\, dx =$$

$$= \left[\frac{x^3}{6} \right]_0^1 = \frac{1}{6}.$$

c) Seja $B = \{(x, y) \mid x^2 + 4y^2 \leqslant 1\}$

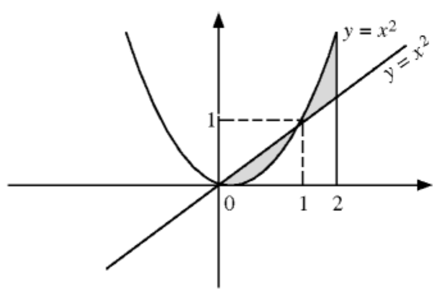
Para cada $x \in [-1, 1]$,

$$\beta(x) = \int_{c(x)}^{d(x)} y\, dy = \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} y\, dy = \left[\frac{y^2}{2}\right]_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} = 0$$



Então, $\iint_B y\, dy\, dx = \int_{-1}^{1} \beta(x)\, dx = 0.$

e) $\iint_B y\, dx\, dy$, onde B é a região compreendida entre os gráficos de $y = x$ e $y = x^2$, com $0 \leqslant x \leqslant 2$. Para cada $x \in [0, 1]$ temos



$$\iint_{B_1} y\,dy\,dx = \int_0^1 \int_{c(x)}^{d(x)} y\,dy\,dx =$$

$$= \int_0^1 \left[\int_{x^2}^{x} y\,dy \right] dx = \int_0^1 \left[\frac{y^2}{2} \right]_{x^2}^{x} dx = \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx =$$

$$= \left[\frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{15}.$$

Para cada $x \in [1, 2]$

$$\iint_{B_2} y\,dy\,dx = \int_1^2 \int_{c_1(x)}^{d_1(x)} y\,dy\,dx = \int_1^2 \left[\int_x^{x^2} y\,dy \right] dx =$$

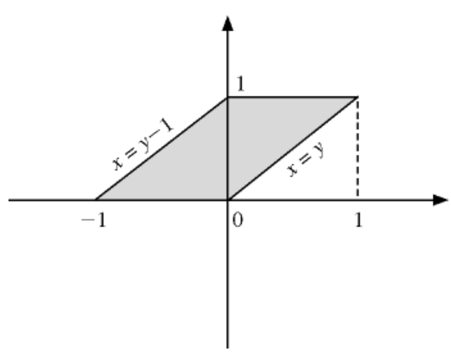
$$= \int_1^2 \left[\frac{y^2}{2} \right]_x^{x^2} dx = \int_1^2 \left(\frac{x^4}{2} - \frac{x^2}{2} \right) dx = \left[\frac{x^5}{10} - \frac{x^3}{6} \right]_1^2 = \frac{29}{15}.$$

Então,

$$\iint_B y\,dy\,dx = \iint_{B_1} y\,dy\,dx + \iint_{B_2} y\,dy\,dx = \frac{1}{15} + \frac{29}{15} = 2.$$

f) $\iint_B y\,dx\,dy$, onde B é o paralelogramo de vértices $(-1, 0)$, $(0, 0)$, $(1, 1)$ e $(0, 1)$.

Para cada y fixo em $[0, 1]$,



$$\iint_B y\,dx\,dy = \int_0^1 \left[\int_{a(y)}^{b(y)} y\,dx \right] dy =$$

$$= \int_0^1 \left[\int_{y-1}^{y} y\,dx \right] dy = \int_0^1 [xy]_{y-1}^{y}\,dy = \int_0^1 y\,dy = \left[\frac{y^2}{2} \right]_0^1 = \frac{1}{2}.$$

Invertendo a ordem de integração,

$$\iint_B y\,dy\,dx = \iint_{B_1} y\,dy\,dx + \iint_{B_2} y\,dy\,dx$$

onde B_1, é o triângulo de vértices $(-1, 0)$, $(0, 0)$ e $(0, 1)$ e B_2 é o triângulo de vértices $(0, 0)$, $(0, 1)$ e $(1, 1)$.

Para cada x fixo em $[-1, 0]$ temos:

$$\int_{-1}^{0}\left[\int_{c(x)}^{d(x)} y\,dy\right]dx = \int_{-1}^{0}\left[\int_{0}^{x+1} y\,dy\right]dx = \int_{-1}^{0} \frac{(x+1)^2}{2}\,dx =$$

$$= \left[\frac{(x+1)^3}{6}\right]_{-1}^{0} = \frac{1}{6}. \quad ①$$

Para cada x fixo em $[0, 1]$ temos

$$\int_{0}^{1}\left[\int_{c_1(x)}^{d_1(x)} y\,dy\right]dx = \int_{0}^{1}\int_{x}^{1} y\,dy\,dx = \int_{0}^{1}\left[\frac{y^2}{2}\right]_{x}^{1} dx =$$

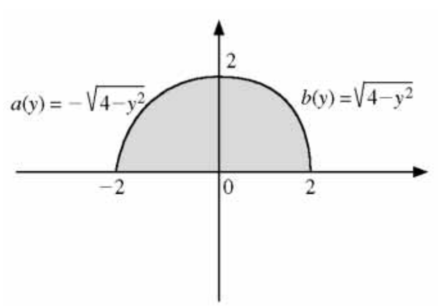
$$= \int_{0}^{1}\left(\frac{1}{2} - \frac{x^2}{2}\right)dx = \left[\frac{1}{2}x - \frac{x^3}{6}\right]_{0}^{1} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \quad ②$$

Então, de ① e ②,

$$\iint_B y\,dy\,dx = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

g) $\iint_B y\,dx\,dy$, onde B é o semicírculo $x^2 + y^2 \leqslant 4$, $y \geqslant 0$.

Para cada y fixo em $[0, 2]$,



$$\alpha(y) = \int_{a(y)}^{b(y)} y\,dx = \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y\,dx$$

ou seja,

$$\alpha(y) = [xy]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} = 2y\sqrt{4-y^2}\,.$$

Então,

$$\iint_B y\,dx\,dy = \int_0^2 \alpha(y)\,dy = \int_0^2 2y\sqrt{4-y^2}\,dy = \left[-\frac{2}{3}(4-y^2)^{3/2}\right]_0^2 = \frac{16}{3}$$

Invertendo a ordem de integração,

para cada x fixo em $[-2, 2]$,

$$\beta(x) = \int_{c(x)}^{d(x)} y\,dy = \int_0^{\sqrt{4-x^2}} y\,dy = \left[\frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} = 2 - \frac{x^2}{2}$$

Então,

$$\iint_B y\,dy\,dx = \int_{-2}^2 \beta(x)\,dx = \int_{-2}^2 \left(2 - \frac{x^2}{2}\right) dx =$$

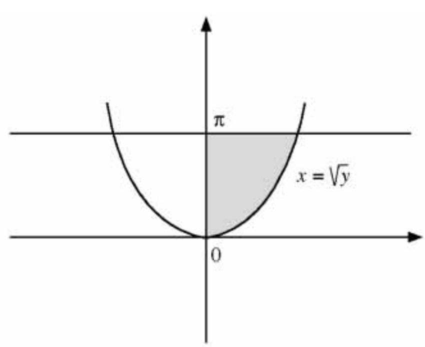
$$= \left[2x - \frac{x^3}{6}\right]_{-2}^2 = \frac{16}{3}.$$

6. *a*) $\iint_B f(x, y)\,dx\,dy$, onde $f(x, y) = x\cos y$ e
$B = \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0, x^2 \leqslant y \leqslant \pi\}$.

Temos $\alpha(y) = \int_{a(y)}^{b(y)} x\cos y\,dx =$

$$= \int_0^{\sqrt{y}} x\cos y\,dx = \cos y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}}, \text{ ou seja,}$$

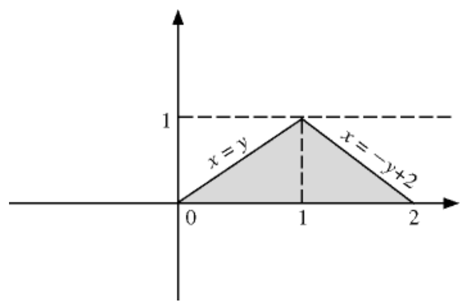
$$\alpha(y) = \frac{y}{2}\cos y.$$



$$\iint_B x \cos y \, dx \, dy = \int_0^{\pi} \alpha(y) \, dy = \int_0^{\pi} \frac{y}{2} \cos y \, dy = \frac{1}{2} \int_0^{\pi} y \cos y \, dy =$$

$$= \frac{1}{2} \left[y \operatorname{sen} y + \cos y \right]_0^{\pi} = \frac{1}{2} (\cos \pi - \cos 0) = -1.$$

c) $\iint_B f(x, y) \, dx \, dy$ onde $f(x, y) = x$ e B é o triângulo de vértices (0, 0), (1, 1) e (2, 0).



$$\iint_B x \, dx \, dy = \int_0^1 \left[\int_{a(y)}^{b(y)} x \, dx \right] dy =$$

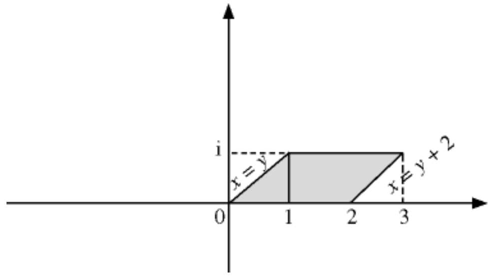
$$= \int_0^1 \left[\int_y^{-y+2} x \, dx \right] dy = \int_0^1 \left[\frac{x^2}{2} \right]_y^{-y+2} dy =$$

$$= \int_0^1 (-2y + 2) \, dy = \left[-y^2 + 2y \right]_0^1 = 1.$$

e) $\iint_B f(x, y) \, dx \, dy$, onde $f(x, y) = x + y$ e B o paralelogramo de vértices (0, 0), (1, 1), (3, 1), (2, 0).

Para cada y fixo em [0, 1],

$$\alpha(y) = \int_{a(y)}^{b(y)} (x + y) \, dx = \int_y^{y+2} (x + y) \, dx = \left[\frac{x^2}{2} + xy \right]_y^{y+2} = 4y + 2.$$



Então,

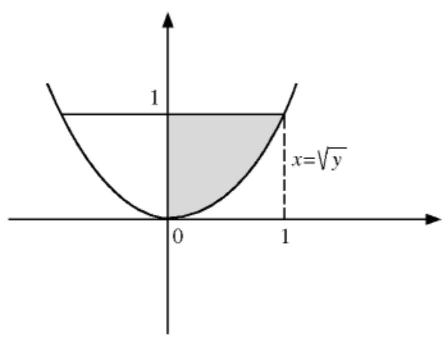
$$\iint_B (x+y)\,dx\,dy = \int_0^1 \left[\int_y^{y+2} (x+y)\,dx\right] dy = \int_0^1 \alpha(y)\,dy =$$

$$= \int_0^1 (4y+2)\,dy = \left[2y^2 + 2y\right]_0^1 = 4.$$

g) $\iint_B f(x, y)\,dx\,dy$, onde $f(x, y) = xy\cos x^2$ e

$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1, x^2 \leqslant y \leqslant 1\}$.

Para y em $[0, 1]$,



$$\alpha(y) = \int_{a(y)}^{b(y)} xy\cos x^2\,dx =$$

$$= \int_0^{\sqrt{y}} xy\cos x^2\,dx = \left[\frac{y}{2}\,\text{sen}\,x^2\right]_0^{\sqrt{y}}, \text{ ou seja,}$$

$$\alpha(y) = \frac{y}{2}\,\text{sen}\,y.$$

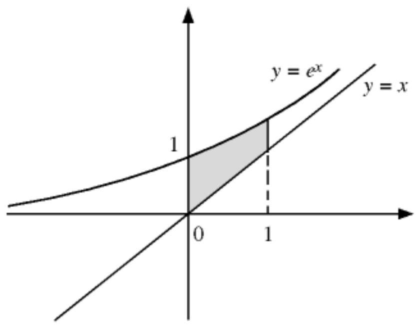
Então,

$$\iint_B f(x, y)\,dx\,dy = \int_0^1 \left[\int_0^{\sqrt{y}} xy\cos x^2\,dx\right] dy =$$

$$= \int_0^1 \alpha(y)\,dy = \int_0^1 \frac{y}{2}\,\text{sen}\,y\,dy = \frac{1}{2}\left[\text{sen}\,y - y\cos y\right]_0^1 =$$

$$= \frac{1}{2}(\text{sen}\,1 - \cos 1).$$

i) $\iint_B f(x, y)\, dx\, dy$, onde $f(x, y) = x + y$ e B é a região compreendida entre os gráficos das funções $y = x$ e $y = e^x$, com $0 \leqslant x \leqslant 1$. Temos

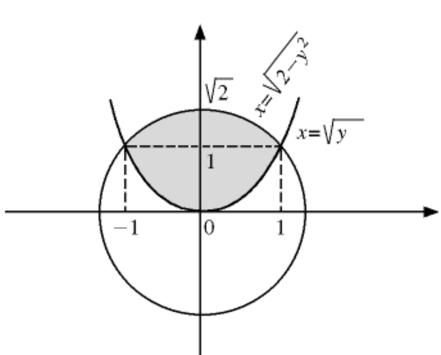


$$\iint_B f(x, y)\, dy\, dx = \int_0^1 \left[\int_{c(x)}^{d(x)} (x + y)\, dy\right] dx = \int_0^1 \left[\int_x^{e^x} (x + y)\, dy\right] dx =$$

$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_x^{e^x} dx = \int_0^1 \left(xe^x + \frac{e^{2x}}{2} - x^2 - \frac{x^2}{2}\right) dx =$$

$$= \int_0^1 \left(xe^x + \frac{e^{2x}}{2} - \frac{3}{2}\, x^2\right) dx = \left[xe^x - e^x + \frac{1}{4}\, e^{2x} - \frac{x^3}{2}\right]_0^1 = \frac{1}{4}\, (1 + e^2).$$

l) $\iint_B f(x, y)\, dx\, dy$, onde $f(x, y) = x^5 \cos y^3$ e
$B = \{(x, y) \in \mathbb{R}^2 \mid y \geqslant x^2,\, x^2 + y^2 \leqslant 2\}$.

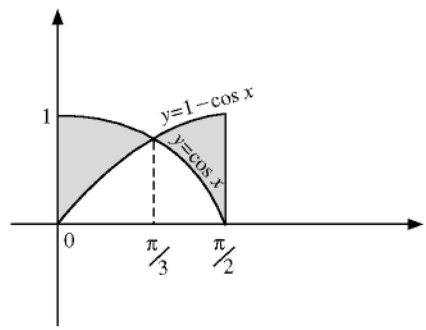


$$\iint_B f(x, y)\, dx\, dy =$$

$$= \int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} x^5 \cos y^3\, dx \right] dy + \int_1^{\sqrt{2}} \left[\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} x^5 \cos y^3\, dx \right] dy = 0,$$

pois $x^5 \cos y^3$ é uma função ímpar na variável x.

n) $\iint_B f(x, y)\, dx\, dy$, onde $f(x, y) = x$ e B é a região compreendida entre os gráficos de $y = \cos x$ e $y = 1 - \cos x$, com $0 \leqslant x \leqslant \dfrac{\pi}{2}$.



$$\iint_B f(x, y)\, dy\, dx =$$

$$= \int_0^{\frac{\pi}{3}} \left[\int_{c(x)}^{d(x)} x\, dy \right] dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\int_{d(x)}^{c(x)} x\, dy \right] dx =$$

$$= \int_0^{\frac{\pi}{3}} \left[\int_{1-\cos x}^{\cos x} x\, dy \right] dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\int_{\cos x}^{1-\cos x} x\, dy \right] dx =$$

$$= \int_0^{\frac{\pi}{3}} [xy]_{1-\cos x}^{\cos x}\, dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} [xy]_{\cos x}^{1-\cos x}\, dx =$$

$$= \int_0^{\frac{\pi}{3}} (2x \cos x - x)\, dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (x - 2x \cos x)\, dx =$$

(lembrando que $\int x \cos x\, dx = x \operatorname{sen} x + \cos x$)

$$= \left[2x \operatorname{sen} x + 2 \cos x - \frac{x^2}{2} \right]_0^{\frac{\pi}{3}} + \left[\frac{x^2}{2} - 2x \operatorname{sen} x - 2 \cos x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} =$$

$$= \frac{\pi^2}{72} + \frac{\pi}{3}(2\sqrt{3} - 3).$$

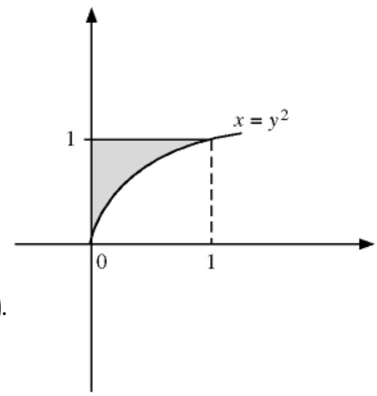
p) $\iint_B f(x, y)\, dx\, dy$, onde $f(x, y) = \sqrt{1 + y^3}$ e $B = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x} \leqslant y \leqslant 1\}$.

$$\iint_B f(x, y)\, dx\, dy = \int_0^1 \int_{a(y)}^{b(y)} \left(\sqrt{1 + y^3}\right) dx\, dy =$$

$$= \int_0^1 \left[\int_0^{y^2} \left(\sqrt{1 + y^3}\right) dx\right] dy =$$

$$= \int_0^1 \left[x\sqrt{1 + y^3}\right]_0^{y^2} dy =$$

$$= \int_0^1 y^2 \sqrt{1 + y^3}\, dy = \frac{2}{9}\left[\left(1 + y^3\right)^{3/2}\right]_0^1 = \frac{2}{9}\left(2\sqrt{2} - 1\right).$$



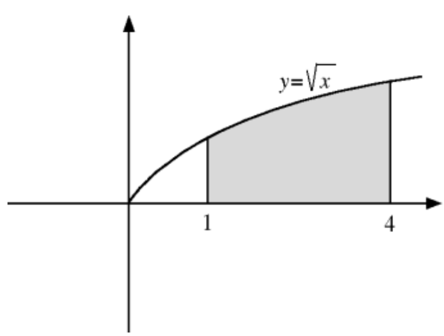
r) $\iint_B f(x, y)\, dx\, dy$, onde $f(x, y) = \dfrac{y}{x + y^2}$ e
$B = \{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x \leqslant 4 \text{ e } 0 \leqslant y \leqslant \sqrt{x}\}$.

Temos

$$\iint_B f(x, y)\, dy\, dx = \int_1^4 \left[\int_0^{\sqrt{x}} \frac{y}{x + y^2}\, dy\right] dx =$$

$$= \int_1^4 \frac{1}{2}\left[\ln\,(x + y^2)\right]_0^{\sqrt{x}} dx = \frac{1}{2}\int_1^4 (\ln 2x - \ln x)\, dx = \frac{1}{2}\int_1^4 \ln 2\, dx =$$

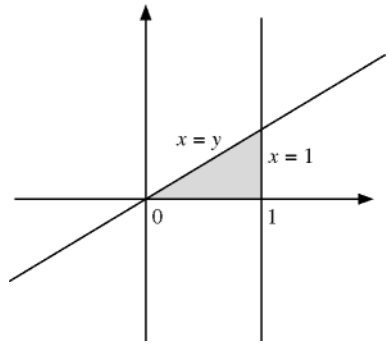
$$= \frac{\ln 2}{2}\,[x]_1^4 = \frac{3}{2}\ln 2.$$



7. *a*) Na integral $\displaystyle\int_0^1 \left[\int_0^x f(x, y)\, dy\right] dx$
o x está variando no intervalo [0, 1] e, para cada x fixo em [0, 1], y varia de 0 a x. A região de integração é:
$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1 \text{ e } 0 \leqslant y \leqslant x\}$.

Então,

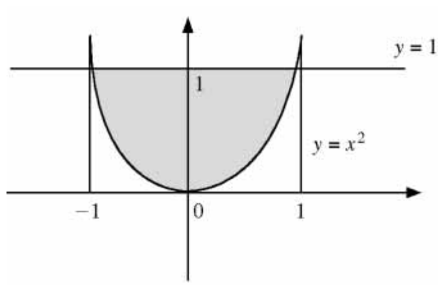
$$\int_0^1 \left[\int_0^x f(x, y)\, dy \right] dx = \int_0^1 \left[\int_y^1 f(x, y)\, dx \right] dy.$$



c) Na integral $\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\, dx \right] dy$

o y está variando no intervalo [0, 1] e, para cada y fixo em [0, 1], x varia de $-\sqrt{y}$ até $\sqrt{y}$.
A região de integração é

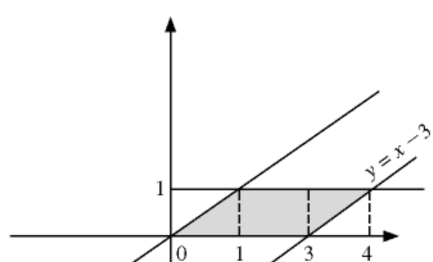
$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant y \leqslant 1, -\sqrt{y} \leqslant x \leqslant \sqrt{y}\}.$$



Então, $\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\, dx \right] dy =$

$$= \int_{-1}^1 \left[\int_{x^2}^1 f(x, y)\, dy \right] dx.$$

e) $\int_0^1 \left[\int_y^{y+3} f(x, y)\, dx \right] dy.$



Para cada y fixo em [0, 1], x varia de y até $y + 3$.

A região de integração é:

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant y \leqslant 1 \text{ e } y \leqslant x \leqslant y + 3\}.$$

Logo,

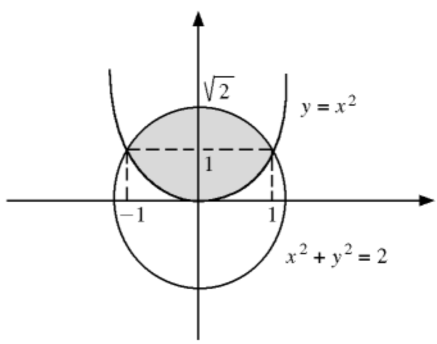
$$\int_0^1\left[\int_y^{y+3} f(x, y)\,dx\right]dy = \int_0^1\left[\int_0^x f(x, y)\,dy\right]dx +$$

$$+\int_1^3\left[\int_0^1 f(x, y)\,dy\right]dx + \int_3^4\left[\int_{x+3}^1 f(x, y)\,dy\right]dx$$

(para cada x fixo em [0, 1], y varia de 0 até x; para cada x fixo em [1, 3], y varia de 0 até 1; para cada x fixo em [3, 4], y varia de $x + 3$ até 1).

g) Na integral $\displaystyle\int_{-1}^1\left[\int_{x^2}^{\sqrt{2-x^2}} f(x, y)\,dy\right]dx$

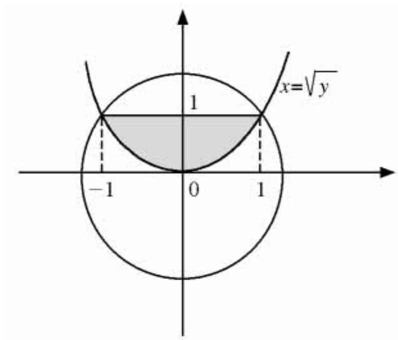
o x está variando em $[-1, 1]$ e, para cada x fixo em $[-1, 1]$, o y varia de x^2 até $\sqrt{2-x^2}$. Seja B a região de integração.

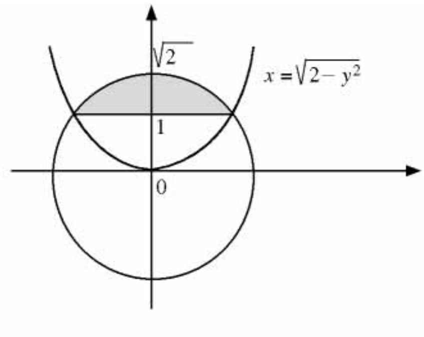


Então B é o conjunto de todos os $(x, y) \in \mathbb{R}^2$ tais que $-1 \leqslant x \leqslant 1$, $x^2 \leqslant y \leqslant \sqrt{2 - x^2}$, ou seja, B é a região do plano compreendida entre os gráficos das funções $y = x^2$ e $y = \sqrt{2 - x^2}$, com $-1 \leqslant x \leqslant 1$.

Temos: $\displaystyle\int_{-1}^1\left[\int_{x^2}^{\sqrt{2-x^2}} f(x, y)\,dy\right]dx = \iint_{B_1} f(x, y)\,dx\,dy + \iint_{B_2} f(x, y)\,dx\,dy$

onde B_1 é o conjunto de todos os (x, y) tais que $0 \leqslant y \leqslant 1$ e $-\sqrt{y} \leqslant x \leqslant \sqrt{y}$ e B_2 é o conjunto de todos os (x, y) tais que $1 \leqslant y \leqslant \sqrt{2}$ e $-\sqrt{2-y^2} \leqslant x \leqslant \sqrt{2 - y^2}$





$$\iint_{B_1} f(x, y)\, dx\, dy = \int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\, dx \right] dy \ \text{ e}$$

$$\iint_{B_2} f(x, y)\, dx\, dy = \int_1^{\sqrt{2}} \left[\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x, y)\, dx \right] dy$$

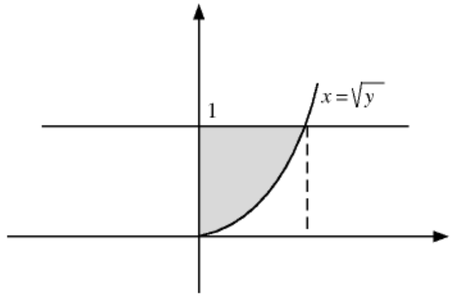
Então,

$$\int_{-1}^1 \left[\int_{x^2}^{\sqrt{2-x^2}} f(x, y)\, dy \right] dx = \int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\, dx \right] dy + \int_1^{\sqrt{2}} \left[\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x, y)\, dx \right] dy.$$

i) Na integral $\int_0^1 \left[\int_{x^2}^1 f(x, y)\, dy \right] dx$ o x está variando em [0, 1] e, para cada x fixo em [0, 1], y varia de x^2 até 1.

Assim, $B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1 \ \text{ e } \ x^2 \leqslant y \leqslant 1\}$

é a região de integração



Ou ainda, $0 \leqslant y \leqslant 1$ e $0 \leqslant x \leqslant \sqrt{y}$.

Então,

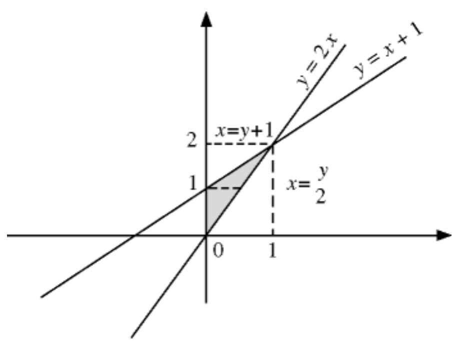
$$\int_0^1 \left[\int_{x^2}^1 f(x, y)\, dy \right] dx =$$

$$= \int_0^1 \left[\int_0^{\sqrt{y}} f(x, y)\, dx \right] dy.$$

l) Na integral $\int_0^1 \left[\int_{2x}^{x+1} f(x, y)\, dy \right] dx,$

a região de integração é

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1, 2x \leqslant y \leqslant x + 1\}$$



Então, $0 \leqslant y \leqslant 1$ e $0 < x < \dfrac{y}{2}$

ou $1 \leqslant y \leqslant 2$ e $y - 1 \leqslant x \leqslant \dfrac{y}{2}$.

Assim,

$$\int_0^1 \left[\int_{2x}^{x+1} f(x, y)\, dy \right] dx = \int_0^1 \left[\int_0^{\frac{y}{2}} f(x, y)\, dx \right] dy +$$

$$+ \int_1^2 \left[\int_{y-1}^{y/2} f(x, y)\, dx \right] dy.$$

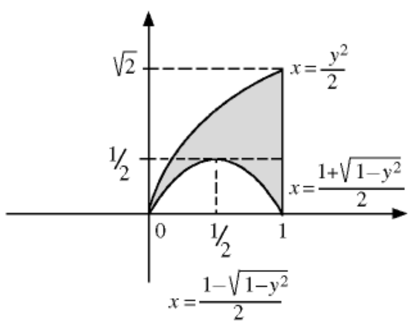
***n*)** Na integral $\int_0^1 \left[\int_{\sqrt{x-x^2}}^{\sqrt{2x}} f(x,y)\,dy \right] dx$

o x está variando em [0, 1] e, para cada x fixo em [0, 1], o y varia de $\sqrt{x - x^2}$ até $\sqrt{2x}$.

Então, B é a região do plano compreendida entre os gráficos das funções $y = \sqrt{2x}$ e $y = \sqrt{x - x^2}$, com $0 \leqslant x \leqslant 1$.

Temos $y = \sqrt{2x} \Rightarrow x = \dfrac{y^2}{2}$

$$y = \sqrt{x - x^2} \Rightarrow -x^2 + x - y^2 = 0 \;\Rightarrow\; x = \frac{1 \pm \sqrt{1-4y^2}}{2}.$$



Então:

$$\int_{-1}^{1} \left[\int_{\sqrt{x-x^2}}^{\sqrt{2x}} f(x,y)\,dy \right] dx =$$

$$= \iint_{B_1} f(x, y)\,dx\,dy + \iint_{B_2} f(x, y)\,dx\,dy + \iint_{B_3} f(x, y)\,dx\,dy$$

onde B_1 é o conjunto de todos os (x, y) tais que $0 \leqslant y \leqslant \dfrac{1}{2}$ e $\dfrac{y^2}{2} \leqslant x \leqslant \dfrac{1 - \sqrt{1 - 4y^2}}{2}$;

B_2 é o conjunto de todos os (x, y) tais que $0 \leqslant y \leqslant \dfrac{1}{2}$ e $\dfrac{1 + \sqrt{1 - 4y^2}}{2} \leqslant x \leqslant 1$ e B_3 é o conjunto de todos os (x, y) tais que $\dfrac{1}{2} \leqslant y \leqslant \sqrt{2}$ e $\dfrac{y^2}{2} \leqslant x \leqslant 1$.

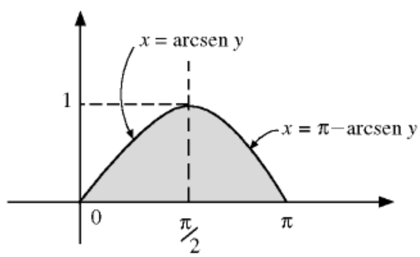
Então,

$$\int_{-1}^{1}\left[\int_{\sqrt{x-x^2}}^{\sqrt{2x}} f(x, y)\, dy\right] dx =$$

$$=\int_0^{\frac{1}{2}}\left[\int_{\frac{y^2}{2}}^{\frac{1-\sqrt{1-4y^2}}{2}} f(x, y)\, dx\right] dy + \int_0^{\frac{1}{2}}\left[\int_{\frac{1+\sqrt{1-4y^2}}{2}}^{1} f(x, y)\, dx\right] dy +$$

$$+\int_{1/2}^{\sqrt{2}}\left[\int_{\frac{y^2}{2}}^{1} f(x, y)\, dx\right] dy.$$

p) Na integral $\int_0^{\pi}\left[\int_0^{\text{sen } x} f(x, y)\, dy\right] dx$ o x está variando em $[0, \pi]$ e, para cada x fixo em $[0, \pi]$, y varia de 0 até sen x.



A região B de integração é

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant \pi \ \text{ e } \ 0 \leqslant y \leqslant \text{sen } x\}.$$

Então,

$$\int_0^{\pi}\left[\int_0^{\text{sen } x} f(x, y)\, dy\right] dx = \int_0^{1}\left[\int_{\text{arcsen } y}^{\pi-\text{arcsen } y} f(x, y)\, dx\right] dy.$$

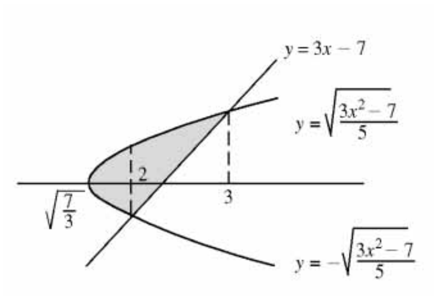
Observe que, para $0 \leqslant x \leqslant \frac{\pi}{2}$ e $0 \leqslant y \leqslant 1$, $y = \text{sen } x \Leftrightarrow x = \text{arcsen } y$; por outro lado, como $\frac{\pi}{2} \leqslant x \leqslant \pi$ é equivalente a $0 \leqslant \pi - x \leqslant \frac{\pi}{2}$ e $y = \text{sen } x = \text{sen}(\pi - x)$, segue que, para $\frac{\pi}{2} \leqslant x \leqslant \pi$ e $0 \leqslant y \leqslant 1$,

$$y = \text{sen } x = \text{sen}(\pi - x) \Leftrightarrow \pi - x = \text{arcsen } y \Leftrightarrow x = \pi - \text{arcsen } y.$$

q) $\int_0^{\pi/4}\int_{\text{sen}\,x}^{\cos x} f(x, y)dydx = \int_0^{\sqrt{2}/2}\int_0^{\text{arcsen}\,y} f(x, y)dxdy + \int_{\sqrt{2}/2}^{1}\int_0^{\arccos y} f(x, y)dxdy.$

r) $x = \dfrac{y+7}{3} \Leftrightarrow y = 3x - 7$ e $x = \sqrt{\dfrac{7+5y^2}{3}} \Leftrightarrow y = \pm\sqrt{\dfrac{3x^2-7}{5}}$. Invertendo a ordem de integração, temos

$$\int_{\sqrt{7/3}}^{2}\int_{-\sqrt{(3x^2-7)/5}}^{\sqrt{(3x^2-7)/5}} f(x, y)dydx + \int_2^3\int_{3x-7}^{\sqrt{(3x^2-7)/5}} f(x, y)dydx$$



8. *a)* $V = \iint_A [4 - (x + y + 2)]\, dx\, dy$, onde A é o círculo $x^2 + y^2 \leqslant 1$.

$$V = \int_{-1}^{1}\left[\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (2 - x - y)\, dx\right] dy = \int_{-1}^{1}\left[2x - \frac{x^2}{2} - xy\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy =$$

$$= \int_{-1}^{1}\left(4\sqrt{1-y^2} - 2y\sqrt{1-y^2}\right) dy = 8\int_0^1 \sqrt{1-y^2}\, dy.$$

(Observe que $\int_{-1}^{1} y\sqrt{1-y^2} = 0,$ pois o integrando é função ímpar e $\int_{-1}^{1}\sqrt{1-y^2}\, dy = 2\int_0^1\sqrt{1-y^2}\, dy,$ pois o integrando é função par.) Fazendo a mudança de variável: $y =$ sen u; $dy = \cos u\, du$, temos

$$V = 8\int_0^{\pi/2}\cos^2 u\, du = 8\left[\frac{1}{4}\,\text{sen}\, 2u + \frac{u}{2}\right]_0^{\pi/2} = 8\cdot\frac{\pi}{4} = 2\pi.$$

$$c)\ V=\int_{-1}^{1}\left[\int_{0}^{1-x^2}(1-x^2)\,dy\right]dx=\int_{0}^{1}\left[y(1-x^2)\right]_0^{1-x^2}\,dx=$$

$$=\int_{-1}^{1}(1-2x^2+x^4)\,dx=\left[x-\frac{2x^3}{3}+\frac{x^5}{5}\right]_{-1}^{1}=\frac{16}{15}.$$

$$e)\ V=2\int_{-2}^{2}\int_{0}^{\sqrt{1-\frac{x^2}{4}}}dy\,dx=2\int_{-2}^{2}\left[y\right]_0^{\sqrt{1-\frac{x^2}{4}}}\,dx=$$

$$=2\int_{-2}^{2}\sqrt{1-\frac{x^2}{4}}=4\int_{0}^{2}\sqrt{1-\frac{x^2}{4}}.$$

Fazendo $\frac{x}{2}=\text{sen }\theta$, $dx=2\cos\theta\,d\theta$.

Temos $x=0\ ;\ \theta=0$

$x=2\ ;\ \theta=\frac{\pi}{2}$.

Então,

$$V=4\int_{0}^{\frac{\pi}{2}}2\cos^2\theta\,d\theta=8\int_{0}^{\frac{\pi}{2}}\cos^2\theta\,d\theta=8\left[\frac{1}{2}\theta+\frac{1}{4}\text{sen }2\theta\right]_0^{\frac{\pi}{2}}=8\left[\frac{\pi}{4}\right]=2\pi.$$

$$g)\ V=2\int_{-a}^{a}\left[\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\sqrt{a^2-y^2}\,dx\right]dy=$$

$$=2\int_{-a}^{a}\left[x\sqrt{a^2-y^2}\right]_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\,dy=$$

$$=4\int_{-a}^{a}(a^2-y^2)\,dy=8\int_{0}^{a}(a^2-y^2)\,dy=$$

$$=8\left[a^2y-\frac{y^3}{3}\right]_0^{a}=8\left(\frac{2a^3}{3}\right)=\frac{16a^3}{3}.$$

$$i)\ V=\int_{0}^{1}\left[\int_{0}^{1-x}(1-x-y)\,dy\right]dx=\int_{0}^{1}\left[y-xy-\frac{y^2}{2}\right]_0^{1-x}\,dx=$$

$$=\int_{0}^{1}\left[(1-x)-x(1-x)-\frac{(1-x)^2}{2}\right]dx=\frac{1}{2}\int_{0}^{1}(1-2x+x^2)\,dx=$$

$$=\frac{1}{2}\left[x-x^2+\frac{x^3}{3}\right]_0^{1}=\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6}.$$

l) Temos $x^2 + y^2 \leqslant z \leqslant 2x$.

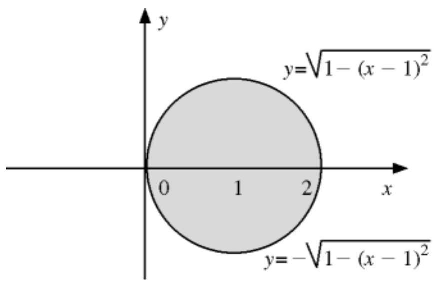
Então, $x^2 + y^2 - 2x \leqslant 0 \Rightarrow (x^2 - 2x + 1) + y^2 \leqslant 1 \Rightarrow (x-1)^2 + y^2 \leqslant 1$

E mais, $0 \leqslant z \leqslant 2x - x^2 - y^2$

$$V = \int_0^2 \left[\int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} (2x - x^2 - y^2)\, dy \right] dx =$$

$$= \int_0^2 \left[2xy - x^2 y - \frac{y^3}{3} \right]_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} dx =$$

$$= \int_0^2 \left[4x\sqrt{1-(x-1)^2} - 2x^2\sqrt{1-(x-1)^2} - \frac{2}{3}\left(\sqrt{1-(x-1)^2}\right)^3 \right] dx.$$



Fazendo $x - 1 = \text{sen}\ \theta$, temos $dx = \cos\theta\, d\theta$.

$x = 0\,;\theta = -\dfrac{\pi}{2}$

$x = 2\,;\theta = \dfrac{\pi}{2}$.

Então,

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[4\,(1 + \text{sen}\ \theta)\cos\theta - 2\,(1 + \text{sen}\ \theta)^2 \cos\theta - \frac{2}{3}\cos^2\theta \right] \cos\theta\, d\theta =$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\cos\theta\,\underbrace{(1 - \text{sen}^2\theta)}_{\cos^2\theta} - \frac{\cos^3\theta}{3} \right] \cos\theta\, d\theta = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta.$$

[Utilizando fórmulas de recorrência (veja Vol. 1; Seção 12.9)

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \cdot \text{sen}\ x + \frac{n-1}{n}\int \cos^{n-2} x\, dx.]$$

Portanto,

$$V = \frac{4}{3}\left\{\left[\frac{1}{4}\cos^3\theta \operatorname{sen}\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta\right\}$$

$$V = \frac{4}{3}\cdot\frac{3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \left[\frac{1}{2}\cos x \operatorname{sen} x + \frac{x}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

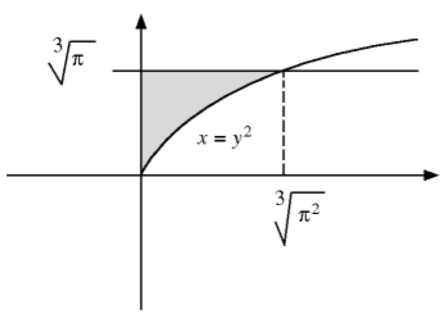
$$V = \left[\frac{1}{4}\operatorname{sen} 2x + \frac{x}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2}.$$

n) $V = \int_0^1\left[\int_0^{1-y}\left[(3x+y+1)-(4x+2y)\right]dx\right]dy =$

$$= \int_0^1\left[\int_0^{1-y}(-x-y+1)\,dx\right]dy = \int_0^1\left[-\frac{x^2}{2} - xy + x\right]_0^{1-y} dy =$$

$$= \frac{1}{2}\int_0^1 (y^2-2y+1)\,dy = \frac{1}{2}\left[\frac{(y-1)^3}{3}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}.$$

o) $\int_0^{\sqrt[3]{\pi}}\int_0^{y^2} \operatorname{sen} y^3\, dx\, dy = \frac{2}{3}.$



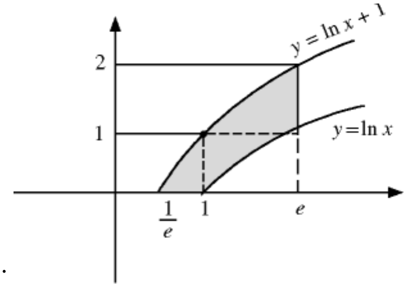
9. *a*) $B = \{(x, y) \in \mathbb{R}^2 \mid \ln x \leqslant y \leqslant 1 + \ln x, y \geqslant 0 \text{ e } x \leqslant e\}.$

Área $= \int_{\frac{1}{e}}^{1}\int_0^{1+\ln x} dy\, dx + \int_1^{e}\int_{\ln x}^{1+\ln x} dy\, dx$

$$\text{Área } = \int_{\frac{1}{e}}^{1} (1 + \ln x)\, dx + \int_{1}^{e} dx$$

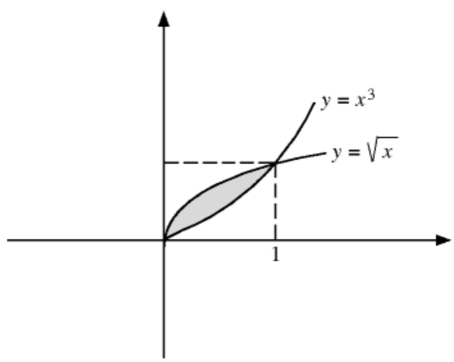
$$\text{Área } = \int_{\frac{1}{e}}^{1} dx + \int_{\frac{1}{e}}^{1} \ln x\, dx + \int_{1}^{e} dx$$

$$\text{Área } = [x]_{1/e}^{1} + [x \ln x - x]_{1/e}^{1} + [x]_{1}^{e} = e + e^{-1} - 1.$$

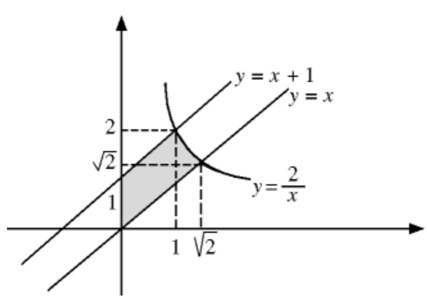


b) $B = \{(x, y) \in \mathbb{R}^2 \mid x^3 \leqslant y \leqslant \sqrt{x}\}$.

$$\text{Área } = \int_{0}^{1} \int_{x^3}^{\sqrt{x}} dy\, dx = \frac{5}{12}.$$



c) $B = \{(x, y) \in \mathbb{R}^2 \mid xy \leqslant 2, x \leqslant y \leqslant x + 1 \text{ e } x \geqslant 0\}$.

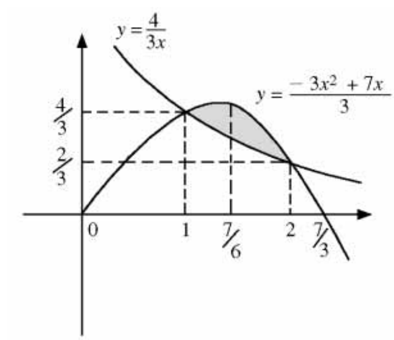


$$\text{Área } = \int_0^1 \int_x^{x+1} dy\, dx + \int_1^{\sqrt{2}} \int_x^{\frac{2}{x}} dy\, dx = \ln 2 + \frac{1}{2}.$$

***d*)** $B = \{(x, y) \in \mathbb{R}^2 \mid x > 0, \frac{4}{x} \leqslant 3y \leqslant -3x^2 + 7x\}.$

Temos

$$\frac{4}{3x} = -\frac{-3x^2 + 7x}{3} \Rightarrow -3x^3 + 7x^2 - 4 = 0$$



Portanto, $x = 1$ e $x = 2$ são abscissas dos pontos de interseção da parábola com a hipérbole.

$$\text{Área } = \int_1^2 \left[\int_{\frac{4}{3x}}^{\frac{-3x^2+7x}{3}} dy \right] dx = \int_1^2 \left[\frac{-3x^2 + 7x}{3} - \frac{4}{3x} \right] dx = \frac{7}{6} - \frac{4}{3} \ln 2.$$

$$\text{Área } = 2 \int_0^2 \left[x - (x^2 - x) \right] dx = \frac{8}{3}.$$